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A Model of Modulated Diffusion. II. Numerical Results on Statistical Properties

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We investigate numerically the statistical properties of a model of modulated diffusion for which we have already computed analytically the diffusion coefficient D. Our model is constructed by adding a deterministic or random noise to the frequency of an integrable isochronous system. We consider in particular the central limit theorem and the invariance principle and we show that they follow whenever D is positive and for any magnitude of the noise; we also investigate the asymptotic distribution in a case when $D = 0$.

KEY WORDS: Decay of correlations; diffusion process.

1. INTRODUCTION

In refs. 1–3 we introduced and studied analytically a model of modulated diffusion; more precisely, we considered an integrable isochronous system whose frequency was successively perturbed by deterministic or purely random noises. What we computed was the diffusion coefficient D for the action variable properly renormalized on the phase space Ω. As for the standard mappings, D is the coefficient of the dominant term in the asymptotic expansion of the variance of a particular stochastic process $S_n(x)$, $x \in \Omega$, that is,

$$\mathbf{E}(S_n^2) = 2Dn(1 + o(1)) \tag{1.1}$$

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where the expectation is taken with respect to the probability measure μ on Ω. The process S_n is related to the evolution of the action variable in time: it is therefore interesting to investigate the statistical properties of a such a process. Among them we will consider in this paper the central limit theorem (CLT) and the invariance principle (IP). In order to establish a CLT for the process S_n it is necessary that D be finite and different from zero; in this case a CLT holds whenever, $\forall z \in \mathbb{R}$,

$$\mu\left(\left\{x \in \Omega; \frac{1}{[\mathrm{E}(S_n^2)]^{1/2}} S_n(x) < z\right\}\right) \xrightarrow[n \to +\infty]{} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{z} e^{-u^2/2}\, du \tag{1.2a}$$

or, equivalently,

$$\mu\left(\left\{x \in \Omega; \frac{1}{(2Dn)^{1/2}} S_n(x) < z\right\}\right) \xrightarrow[n \to +\infty]{} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{z} e^{-u^2/2}\, du \tag{1.2b}$$

A strong refinement of the CLT is the so-called invariance principle, which consists in redefining the process $S_{[nt]}$,[4] without changing its distribution, on a new probability space with standard Brownian motion $(W(t), t \geq 0)$ and states that

$$\frac{S_{[nt]}}{(2Dn)^{1/2}} \xrightarrow[n \to +\infty]{\text{weakly}} W(t) \tag{1.3}$$

A more direct way to understand (1.3), which also gives the prescriptions to check it numerically, is that after a rescaling of the discrete time $n \to [nt]$ and for μ-almost all $x \in \Omega$, the process $S_{[nt]}(x)/(2Dn)^{1/2}$ is, as a function of t, approximately distributed for n large as the path up to time $t = 1$ of a Brownian particle (see refs. 4–7 and ref. 8 for applications of this idea). The possibility to check conditions (1.2) and (1.3) in our cases relies on the structure of the process $S_n(x)$, which, as we will show in Section 2, can be written as

$$S_n(x) = \sum_{j=0}^{n-1} \eta_j(\theta, \alpha) \tag{1.4}$$

where x is now the couple $(\theta, \alpha) \in \mathbb{T} \times B = \Omega$, where $\mathbb{T}$ is the one-dimensional torus parametrized by $[0, 2\pi[$, B is a compact subset of $\mathbb{R}$, usually the unit interval or again the unit circle, and η_j is an analytic observable on Ω. We have already said that our model is obtained by perturbing an integrable system: the perturbation is deterministic or purely random and

[4] Here we denote by $[y]$ the integer part of the real number y.

in both cases we equip Ω with a probability measure μ in such a way that the sequence of functions $\eta_j(\theta, \alpha)$ is a stochastic process on Ω with the common distribution μ (as we will see in Section 2, μ has a product structure, factorized on the spaces $\mathbb{T}$ and B). Now it turns out that when the perturbation is deterministic, the stochastic process $\eta_j(\theta, \alpha)$ is stationary. This is not generally the case when the modulation is random. In the latter case the diffusion coefficient will not generally satisfy the asymptotic expression (1.1), although it is still given by

$$D = \lim_{n \to +\infty} \frac{1}{2n} \mathbf{E}(S_n^2) \tag{1.5}$$

We now return to the deterministic case because the statistical properties are, at least in principle, easier to understand. It will follow from Section 2 that in this case

$$\eta_j(\theta, \alpha) = g(T^j(\theta, \alpha)) \tag{1.6}$$

where g is an analytic function of both its arguments. The mapping T has the structure of a skew system:

$$T: \quad \begin{cases} \alpha' = T_B(\alpha) \\ \theta' = T_A(\theta) + \varepsilon f(\alpha) \quad \mathrm{mod}[0, 2\pi[\end{cases} \tag{1.7}$$

where T_A is the irrational rotation, leaving invariant the Lebesgue measure μ_A, T_B is a smooth transformation of the unit interval or the unit circle, leaving invariant the probability measure μ_B, f is a smooth function on B of order 1, and ε is the perturbation parameter. The mapping T leaves invariant the product measure $\mu = \mu_A \times \mu_B$, so that, by (1.6), the process η_j is stationary with respect to μ, as anticipated before. The structure (1.4) of the process S_n suggests that a CLT should follow from some sort of independence of the random variables η_j. We expect this is true whenever the strength of the perturbation increases, according to the well-known random phase approximation[9]: in this case the diffusion coefficient should converge to its quasilinear value, too. This last fact was effectively checked by us for almost all the modulations considered in ref. 1. What emerges from the numerical simulations presented in this paper is that also at intermediate values of ε, for which the random variables η_j are surely not independent, a CLT is still valid. This means that some weak independence of the variables η_j continues to persist and we guess this is a consequence of the ergodic properties of the mapping T, in particular of the factor T_B. Before illustrating this point, we emphasize that T is constructed by coupling T_B with a rotation and this makes very difficult to prove analytically a CLT even if T_B satisfies strong Markov or Bernoulli properties (although for arbitrary aperiodic dynamical systems, including ergodic rotations of the circle, the existence of some particular observables satisfying a CLT has been recently proved.[10]) When $T_B(\alpha)=2\alpha \bmod[0,1[$ and $\varepsilon/2\pi \in \mathbb{R}\backslash\mathbb{Q}$, it is possible to show[11] that the skew system (Ω, T, μ) is mixing. We suspect that stronger properties hold, for example, exactness, which should guarantee those conditions of Rosenblatt (φ-mixing) or Ibragimov type sufficient to establish a CLT for the sequence $\eta_1+\eta_2+\cdots$. Our guess is also supported by the computation of D itself: in fact the limit (1.5) is equivalent to the so-called discrete Green–Kubo formula:

$$D=\frac{1}{2}\mathbf{E}(\eta_0^2)+\sum_{j=1}^{\infty} \mathbf{E}(\eta_0\eta_j) \tag{1.8}$$

The finiteness of the sum in (1.8) implies a fast decay of the correlation functions $\mathbf{E}(\eta_0\eta_j)$, and this is just an indication of strong mixing properties of the system. As a completely different case, we will show that when T_B is itself a rotation of the circle, then there is a large class of processes η_j for which the CLT is violated. In these cases we already proved in ref. 2 that the diffusion coefficient is zero, so that the variance of S_n is degenerate in the limit $n \to +\infty$.

The plan of the paper is the following: in Section 2 we recall the definition and the main properties of our model; in Section 3 we illustrate the algorithms to check the CLT and we report the numerical results in the presence of diffusion: in this case we effectively show that a central limit theorem holds; we also look for a CLT when the transport is absent and $D=0$: the limit (1.2) will recover, in this case, a sort of bimodal distribution. Finally, in Section 4 we explain how to check the invariance principle and discuss the numerical results we found; the conclusions are in Section 5.

2. A REVIEW OF THE MODEL

According to Section 2.1 of ref. 2 (hereafter referred to as Part I), we consider the modulated map on the cylinder $(\theta, j) \in \mathbb{T}\times\mathbb{R}=[0, 2\pi[\times\mathbb{R}$:

$$M_p\colon\quad \begin{cases} \theta_{n+1}=\theta_n+\omega+\varepsilon f(\alpha_n) \mod[0,2\pi[\\ j_{n+1}=j_n+V(\theta_n) \end{cases} \tag{2.1}$$

The noise is $\varepsilon f(\alpha_n)$, where f is a smooth function of its argument and α_n is defined in the following way: either it is a deterministic dynamical system generated by some transformation T_B acting on the space B of the initial conditions α_0 and leaving invariant the measure μ_B, that is, $\alpha_n := \alpha_n(\alpha_0) = T_B^n(\alpha_0)$, or it is a stochastic process $\alpha_n(\alpha_0)$ on B, where α_0 belongs to some probability space B (we still use the same letter), with probability measure μ_B (we sometimes drop the explicit dependence of α_n on α_0). In both cases the strength of the perturbation is of order ε: when the perturbation is small, our model should describe the transport in regions of slowly varying frequency; we return to this point in Section 5. It is convenient to describe the transport properties of M_p in terms of the new action J defined by Eq. (2.1.2) of Part I and consider the variation ΔJ_{n+1} given by (2.2.2) of Part I as the stochastic process S_n introduced in Section 1; the random variable $\eta_j(\theta_0, \alpha_0)$ on the space of initial data θ_0, α_0 is now given by the inner sum in (2.2.2) Part I and in the deterministic case it can be written also in the form (1.6). Still in the deterministic case, the quantity ΔJ_{n+1} can be viewed as an observable on the invariant set of the mapping (1.7), where T_A is now the rotation $T_A(\theta) = \theta + \omega \bmod [0, 2\pi[$. All the arguments presented in the Introduction can therefore be translated to ΔJ_{n+1}. We now specialize the process $\alpha_n(\alpha_0)$ for the purposes of this paper, and recall the results found for the diffusion coefficient D of the corresponding process ΔJ_{n+1}.

(a) $T_B(\alpha) = 2\alpha \bmod [0, 1[- \frac{1}{2}$, $\alpha \in [-\frac{1}{2}, \frac{1}{2}[$, which is a Markov map of the unit circle $B = [-\frac{1}{2}, \frac{1}{2}[$ with respect to the Lebesgue measure $\mu_B = \mu_L$ and $f(\alpha) = \alpha$, $\forall \alpha \in [-\frac{1}{2}, \frac{1}{2}[$. In this case the limit variance $\sigma^2 = 2D$ exists independently of J_0 according to formula (2.3.1) of Part I and a direct numerical computation also shows that $\sigma^2 > 0$ whenever $\varepsilon \neq 0$.

(b) $T_B(\alpha) = \sigma(\alpha)$, where $\alpha = \{\omega_i\}_{i=-\infty}^{\infty}$ belongs to the space A of biinfinite sequences at values in the set $X = \{s_1, s_2, ..., s_M\}$, $s_i \in \mathbb{R}$, $1 \leqslant i \leqslant M$, endowed with the discrete topology. The set of probabilities $\mu\{s_i\} = p_i$, $0 < p_i < 1$, $\sum_{i=1}^{M} p_i = 1$ defines a measure $\bar{\mu}$ on A through $\bar{\mu} = \bigotimes_{i=-\infty}^{\infty} \mu$, invariant with respect to the transformation σ acting on A as the shift $(\sigma\alpha)_i = \alpha_{i+1}$. We take $f(\alpha) = \omega_0$, the projection of the word α onto the zeroth coordinate. The previous prescription is equivalent to introducing a random noise of type $\varepsilon\alpha_n$, where $\{\alpha_n\}_{n\in\mathbb{N}}$ is an i.i.d. stochastic process whose variables α_n take the real values $s_1, s_2, ..., s_M$ with probabilities $p_1, p_2, ..., p_M$, respectively. For $M = 2$ the limit variance can be written in a form independent of J_0, by setting $Q_k = p_1 \exp(ik\varepsilon s_1) + p_2 \exp(ik\varepsilon s_2)$:

$$\sigma^2 = \mathrm{Re}\left\{\sum_{k\neq 0} |V_k|^2 \left[1 + \frac{2Q_k e^{ik\omega}}{1 - Q_k e^{ik\omega}}\right]\right\} > 0 \tag{2.2}$$

(c) $T_B(\alpha) = \alpha + \rho \bmod [0, 2\pi[$, with $\rho/2\pi \in \mathbb{R}\backslash\mathbb{Q}$, an irrational rotation, ergodic with respect to the Lebesgue measure μ_L on $\mathbb{R}$. The function

f is chosen as $f(\alpha)=\cos\alpha$, $\forall\alpha\in\mathbb{R}$. Owing to the persistence of invariant surfaces in the extended phase space $(\alpha, \theta, j)\in[0, 1[\times[0, 2\pi[\times\mathbb{R}$, no diffusion in the action variable occurs, so that $\sigma^2=0$.

(d) We finally consider a perturbation of stochastic type $\varepsilon\alpha_n$, where α_n is the realization of an i.i.d. real stochastic process $\alpha_n(\alpha_0)$, $\alpha_0\in B$, on some probability space (B, μ_B) and the distribution of μ_B is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. The function f is simply the identity. The general expression for the finite and positive diffusion coefficient is specialized in Section 3 for two cases of particular interest, the uniform distribution and the Gaussian one.

3. CENTRAL LIMIT THEOREM

In order to check the central limit property of our models we simply apply the definition and estimate, for large $n\in\mathbb{N}$, the probability distribution of the random variable

$$\Xi_n(\theta, \alpha)=\frac{J_n-J_0}{\sqrt{n}\,\sigma}=\frac{1}{\sqrt{n}\,\sigma}\sum_{j=0}^{n-1} g(T^j(\theta, \alpha)) \tag{3.1}$$

where $\sigma=(2D)^{1/2}>0$ and $(\theta, \alpha)\in\Omega=\mathbb{T}\times B$.

Equivalently, on having defined $\sigma_n^2>0$ as the variance of the random variable $(J_n-J_0)/\sqrt{n}$, we can consider the stochastic process

$$H_n(\theta, \alpha)=\frac{J_n-J_0}{\sqrt{n}\,\sigma_n} \tag{3.2}$$

A simple calculation shows that, whenever $\sigma_n>0$, $\forall n\in\mathbb{N}$, and $\lim_{n\to+\infty}\sigma_n^2=\sigma^2>0$, the random variable Ξ_n converges in distribution to the normal variable when $n\to+\infty$ if and only if H_n satisfies the same property; however, as the variance of H_n is normalized to 1, we may expect that the convergence of H_n is faster than that of Ξ_n.

The central limit theorem means that $\forall[a, b]\subseteq\mathbb{R}$ there holds

$$\lim_{n\to+\infty}\mu\left(\left\{(\theta, \alpha)\in\Omega; a\leqslant\Xi_n(\theta, \alpha)\leqslant b\right\}\right)=\frac{1}{(2\pi)^{1/2}}\int_{[a,b]} e^{-\lambda^2/2}\,d\lambda \tag{3.3}$$

To check this we can choose an interval $[-c, c[\subseteq\mathbb{R}$ and divide it into a suitable number of equal subintervals $[a_i, b_i[$, $i=1, 2,..., N$, $b_i=a_{i+1}$, $a_1=-c$, $b_N=c$; for each interval $[a_i, b_i[$ we give an estimate of the measure μ of the set:

$$\{(\theta, \alpha)\in\Omega; a_i\leqslant\Xi_n(\theta, \alpha)<b_i\} \tag{3.4}$$

by means of Monte Carlo techniques, and compare it with the integral of the normal distribution on the same interval. Moreover, a piecewise constant fit of the probability distribution of Ξ_n on the interval $[-c, c[$ (at fixed n) can be simply written as

$$\rho(y) = \sum_{i=1}^{N} \frac{\mu(\{(\theta, \alpha) \in \Omega; a_i \leqslant \Xi_n(\theta, \alpha) < b_i\})}{b_i - a_i} \cdot \mathscr{X}_{[a_i, b_i[}(y) \tag{3.5}$$

The observable $\Xi_n(\theta, \alpha)$ can be written in the form $\Xi_n(\theta, \alpha) = \sum_{j=0}^{n-1} g(\theta_j, \alpha_j)/(\sqrt{n}\,\sigma)$, where $(\theta_j, \alpha_j) = T^j(\theta, \alpha)$, $(\theta_0, \alpha_0) = (\theta, \alpha)$, and

$$g(\theta, \alpha) = \sum_{k \neq 0} \frac{V_k e^{ik(\omega + \theta)}}{e^{ik\omega} - 1} [1 - e^{ik\varepsilon f(\alpha)}] \tag{3.6}$$

We distinguish some different cases according to the features of the frequency modulation,[2] as already mentioned in the previous section.

Case (a). The system takes a form more suitable for computations by means of the change of variables $\beta = \alpha + 1/2$, $\phi = \theta/2\pi$, $(\phi, \beta) \in [0, 1[^2$, which leads to

$$\Xi_n(\phi, \beta) = \frac{1}{\sqrt{n}\,\sigma} \sum_{j=0}^{n-1} \hat{g}(\phi_j, \beta_j) \tag{3.7}$$

with

$$\hat{g}(\phi, \beta) = \sum_{k \neq 0} \frac{V_k e^{ik(\omega + 2\pi\phi)}}{e^{ik\omega} - 1} [1 - e^{ik\varepsilon(\beta - 1/2)}] \tag{3.8}$$

and, $\forall j \in \mathbb{N}$,

$$\hat{T}: \begin{cases} \beta_j = 2\beta_{j-1} \bmod [0, 1[\\ \phi_j = \phi_{j-1} + \dfrac{1}{2\pi}\left(\omega - \dfrac{\varepsilon}{2}\right) + \dfrac{\varepsilon}{2\pi}\beta_{j-1} \quad \bmod [0, 1[\end{cases} \tag{3.9}$$

Because of the strong expansivity of $T_B(\beta) = 2\beta \bmod [0, 1[$, which gives rise to meaningless results after a few iterations, the numerical computation of the trajectory in the phase space $\Omega = [0, 1[^2$ of an initial point $(\phi, \beta) \in \Omega$ cannot be performed directly and the use of symbolic dynamics is needed. If we want to compute the trajectory up to order $n \in \mathbb{N}$, the initial datum $\beta \in [0, 1[$ can be represented in a suitable way by the finite sequence $(b_1, b_2,..., b_n, b_{n+1},..., b_m)$, where $m \gg n$ and $\beta = \sum_{j=1}^{\infty} b_j 2^{-j}$ is the binary representation of the real number $\beta \in [0, 1[$. The action of T_B on β is then equivalent to the left shift on the sequence of b_i and it is easy to find the following relationships for the nth iterate through $\hat{T}$ of the initial point $(\phi, \beta) \in [0, 1[^2$:

$$\begin{cases} \beta_n = \sum_{j=1}^{m-n} b_{j+n} \dfrac{1}{2^j} \\ \phi_n = \phi + n \dfrac{1}{2\pi}\left(\omega - \dfrac{\varepsilon}{2}\right) + \dfrac{\varepsilon}{2\pi} \sum_{j=0}^{n-1} \beta_j \quad \mathrm{mod}[0, 1[\end{cases} \tag{3.10}$$

with $\beta_0 = \beta$ and $\sum_{j=0}^{n-1} \beta_j = \sum_{j=1}^{n} b_j + \beta_n - \beta$. For numerical purposes a satisfactory precision can be obtained by setting $m = n + 30$.

A Monte Carlo estimate of (3.4) requires a random generation of initial points (ϕ, β) with uniform distribution in the unit square $[0, 1[^2$, which is equivalent to employing two independent random number generators for $\beta \in [0, 1[$ and $\phi \in [0, 1[$; a uniform distribution of random points $\beta \in [0, 1[$ can be easily achieved by assigning independently the random values 0, 1, with constant probability 1/2, to each bit $b_1, b_2, ..., b_m$ of the binary representation of β.

For simplicity we consider the potential $V(\theta) = \sqrt{2} \sin\theta$, for which the function $\hat{g}(\phi, \beta)$ turns out to be

$$\hat{g}(\phi, \beta) = \frac{1}{\sqrt{2}\sin(\omega/2)} \left[\cos\left(2\pi\phi + \frac{\omega - \varepsilon}{2} + \varepsilon\beta\right) - \cos\left(2\pi\phi + \frac{\omega}{2}\right)\right] \tag{3.11}$$

and the limit variance can be easily computed.[2]

Figure 1 shows the piecewise constant estimate of the probability distribution ρ of $\Xi_n(\phi, \beta)$ at time $n = 600$ for $\omega/2\pi = (\sqrt{5} - 1)/2$ (the golden mean) and $\varepsilon = 16.137$. For this value of ε the diffusion coefficient D is significantly less than the quasilinear estimate 1. The probability distribution is scanned over the interval $[-c, c[= [-3.4, 3.4[$, partitioned into 200 subintervals of equal amplitude. The superposed, smooth curve is the normal distribution: the diagram suggests a good agreement between the two distributions, even for quite poor statistics (we sample the phase space $[0, 1[^2$ by 900,000 uniformly distributed random points only).

An analogous conclusion can be deduced from Fig. 2, where the numerical distribution is represented for the case $\varepsilon = 100.123$ over the interval $[-c, c[= [-3, 3[$. With this choice of the coupling parameter ε the diffusion coefficient turns out to be[2] close to the quasilinear estimate.

Case (b). For simplicity we can confine ourselves to the case $M = 2$ and consider the initial sequence $\alpha_0 = (\omega_0, \omega_1, \omega_2, ...)$, where $\omega_j \in \{s_1, s_2\} \subset \mathbb{R}$, $\forall j \in \mathbb{N} \cup \{0\}$, and the symbols s_1, s_2 occur with probability ρ_1 and ρ_2,

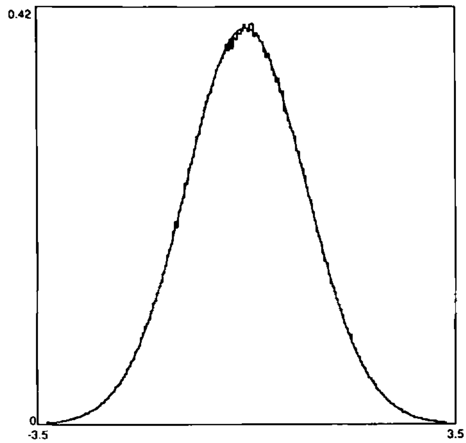

Fig. 1. Probability distribution for the case of the Markov modulation $2\beta \bmod [0, 1[$ after $n = 600$ iterations of the mapping M_p, with parameters $\omega/2\pi = (\sqrt{5} - 1)/2$ (the golden mean) and $\varepsilon = 16.137$. The distribution is estimated over 200 bins of the interval $[-3.4, 3.4[$ by means of 900,000 random initial conditions. The normal distribution (smooth line) is superposed.

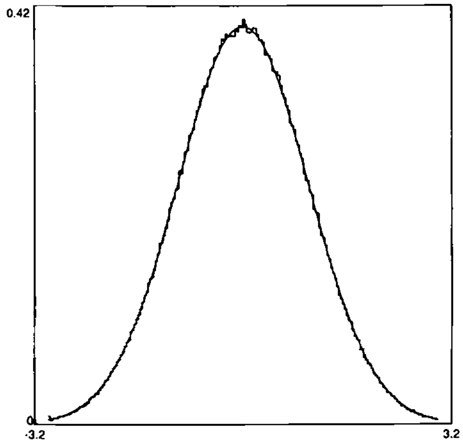

Fig. 2. The same as in Fig. 1, with $\varepsilon = 100.123$, on the interval $[-3, 3[$. The mapping M_p satisfies the random phase approximation.

respectively, $\rho_1, \rho_2 \in [0, 1[$, $\rho_1 + \rho_2 = 1$. By setting $(\theta_j, \alpha_j) = T^j(\theta, \alpha)$, $\forall j \in \mathbb{N}$, $\alpha_j = (\omega_j, \omega_{j+1},...)$, the stochastic process to consider is then

$$\Xi_n(\theta, \alpha) = \frac{1}{\sqrt{n}\,\sigma} \sum_{j=0}^{n-1} g(\theta_j, \alpha_j) \tag{3.12}$$

with the limit variance (2.2). For the particular case $V(\theta) = \sqrt{2} \sin \theta$ the function g takes the explicit form

$$g(\theta, \alpha) = \frac{1}{\sqrt{2}\sin(\omega/2)} \left[\cos\left(\theta + \frac{\omega}{2} + \varepsilon f(\alpha)\right) - \cos\left(\theta + \frac{\omega}{2}\right) \right] \tag{3.13}$$

Notice that, if $\alpha_0 = (\omega_0, \omega_1, \omega_2,...)$, then by definition $f(\alpha_n) = \omega_n$, $\forall n \in \mathbb{N}$. The random choice—with uniform distribution—of the initial sequence α_0 is therefore equivalent to a random, stochastically independent generation of the symbols $\omega_0, \omega_1,..., \omega_n \in \{s_1, s_2\}$, provided that the values s_1 and s_2 occur with probabilities ρ_1 and ρ_2.

The result of a numerical computation of the probability distribution for various choices of the parameters is shown in Fig. 3.

Case (c). By setting again $V(\theta) = \sqrt{2} \sin \theta$, we consider the stochastic process

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g(\theta_j, \alpha_j) \tag{3.14}$$

with $g(\theta, \alpha)$ as in (3.13). By means of perturbative techniques it is possib to prove[2] that the limit variance of (3.14), as $n \to +\infty$, is zero, so that v

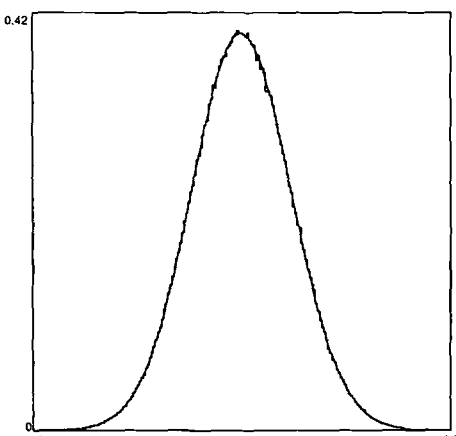

Fig. 3. Probability distribution for the case of the Bernoulli noise, after $n = 600$ iterations of M_p, with parameters $\omega/2\pi = (\sqrt{5} - 1)/2$ and $\varepsilon = 10.137$. The distribution is estimated over 200 bins of the interval $[-4, 4[$ by means of 950,000 random initial conditions.

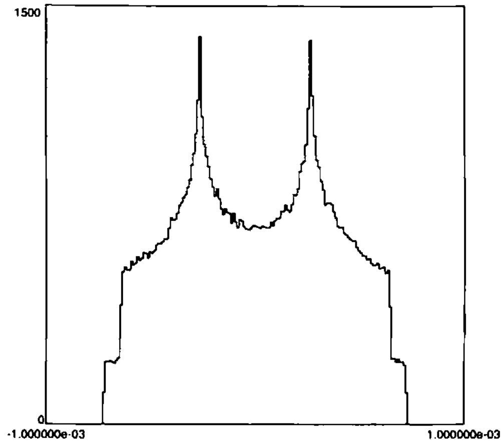

Fig. 4. Probability distribution at time $n = 1000$ in the case of the quasiperiodic modulation. The parameters are $\omega = 2\pi(\sqrt{5} - 1)/2$, $\rho = 2\pi/4$, $\varepsilon = 10.137$. Estimated by means of 640,000 initial conditions with 200 bins in the interval $[-10^{-3}, 10^{-3}[$.

cannot expect that a CLT holds in this case. The probability distribution shown in Fig. 4 for the case $\omega = 2\pi(\sqrt{5} - 1)/2$, $\rho = 2\pi/4$, with $\varepsilon = 10.137$, confirms that there is no weak convergence to a normal distribution. A more interesting problem might consist in replacing (3.14) by the unit-variance process $\sum_{j=0}^{n-1} g(\theta_j, \alpha_j)/\sigma_n \sqrt{n}$, where $\sigma_n^2 > 0$ is the variance of the random variable (3.14). Numerical investigations suggest the existence of a bimodal limit distribution for (3.14) as $n \to +\infty$. Such a distribution follows from the projection along J of the invariant surface supporting the dynamics in the extended phase space $(\alpha, \theta, j) \in [0, 1[\times [0, 2\pi[\times \mathbb{R}$.

Case (d). For the potential $V(\theta) = \sqrt{2} \sin \theta$ the stochastic process to consider is

$$\Xi_n(\theta_0, \alpha_0, \alpha_1, ..., \alpha_{n-1}) = \frac{1}{\sqrt{n}\,\sigma} \sum_{j=0}^{n-1} g(\theta_j, \alpha_j) \tag{3.15}$$

with $g(\theta, \alpha)$ given again by (3.13) and the limit variance σ^2 defined in terms of the diffusion coefficient as usual through $\sigma^2 = 2D$.

Two cases of particular interest have been investigated. First we consider the case of independent random variables with uniform[5] distribution $\theta(r - |\alpha - a|)/2r$, for which the limit variance takes the form[2]

$$\sigma^2 = \frac{1}{2\,|\sin(\omega/2)|^2} \operatorname{Re} \left\{ 1 - \mathscr{X}(\varepsilon) - e^{i\omega}(1 - \mathscr{X}(\varepsilon))^2 \frac{1}{1 - e^{i\omega}\mathscr{X}(\varepsilon)} \right\} \tag{3.16}$$

[5] Here θ denotes the usual Heaviside step function.

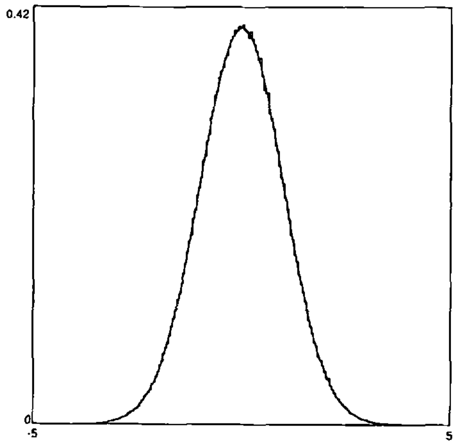

Fig. 5. Probability density at $n = 600$ for an i.i.d. random process, with uniform distribution $\theta(r - |\alpha - a|)/2r$, $\alpha = 0.5$, $r = 0.2$. Here ω is again the golden mean, whereas $\varepsilon = 10.137$. We took 1,100,000 initial points with 200 bins on the interval $[-4, 4[$.

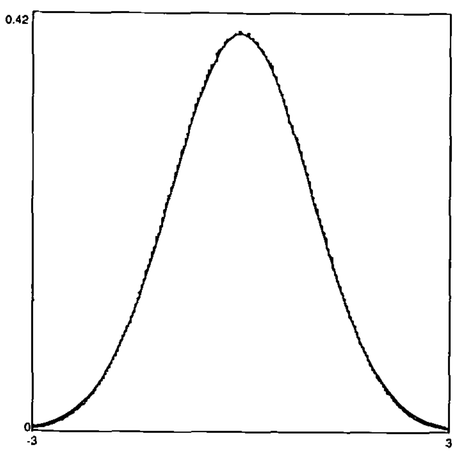

Fig. 6. Probability density after $n = 10$ steps on the interval $[-3, 3[$ for the case of an i.i.d. random process with Gaussian distribution of zero mean and standard deviation 1. Here $\varepsilon = 10.137$, with 200 bins, and 4,000,000 initial conditions.

with $\mathscr{X}(\varepsilon) = e^{i\varepsilon a} \sin(\varepsilon r)/\varepsilon r$. Figure 5 illustrates the result of a Monte Carlo simulation performed for the value of the coupling parameter $\varepsilon = 10.137$; the piecewise constant probability distribution agrees very well with the normal one. Another interesting situation arises when the i.i.d. random variables are Gaussian, with mean value $a \in \mathbb{R}$ and standard deviation $\lambda > 0$. In this case the same expression (3.16) for the limit variance holds, with $\mathscr{X}(\varepsilon) = \exp(ia\varepsilon - \lambda^2\varepsilon^2/2)$. A good agreement with the normal law is achieved even by considering a small number of iterations of the perturbed mapping, as suggested by Fig. 6, $n = 10$.

4. INVARIANCE PRINCIPLE

The invariance principle, as illustrated in the Introduction, states the capability of the system to yield realizations of the process $S_{[nt]}(x)/(2Dn)^{1/2}$, trajectories on the real line parametrized by the continuous time $t \in \mathbb{R}^+$, with a probability distribution close to that of a Wiener process in the limit of large $n \in \mathbb{N}$. In other words, a system for which an invariance principle holds can be viewed as a good "generator of Brownian motion." Therefore, it is quite reasonable to use realizations of the above process $S_{[nt]}(x)/(2Dn)^{1/2}$ in order to investigate the validity of an IP for the class of models described here; we simply have to check if an arbitrary trajectory of $S_{[nt]}(x)/(2Dn)^{1/2}$ can be considered as the result of a Brownian motion, that is, the realization of a Wiener process. To this end a very useful and standard technique[12] consists in the computation of the so-called power spectrum of the process. Among the advantages of such a method are the fast implementation and the possibility to obtain good results even by using a unique realization of the process. Denoting by $X(t)$, $t \in \mathbb{R}^+$, a realization of the process $S_{[nt]}(x)/(2Dn)^{1/2}$ for a large, fixed $n \in \mathbb{N}$, and by $X(t; \tau)$ the restriction of $X(t)$ up to time $\tau > 0$, let us consider the Fourier transform

$$F(\nu, \tau) := \int_{[0, \tau]} X(t)\, e^{-i2\pi\nu t}\, dt, \qquad \nu \in \mathbb{R} \tag{4.1}$$

The spectral density $\mathscr{P}(\nu)$ of $S_{[nt]}(x)/(2Dn)^{1/2}$ can then be obtained by means of the relationship

$$\mathscr{P}(\nu) = \lim_{\tau \to +\infty} \frac{1}{\tau} |F(\nu; \tau)|^2 \tag{4.2}$$

Good estimates of (4.2) can be achieved by taking the time $\tau \in \mathbb{R}^+$ large enough and computing $F(\nu; \tau)$ by means of a fast Fourier transform algorithm. Whenever $X(t)$ is the realization of a Wiener process the dependence

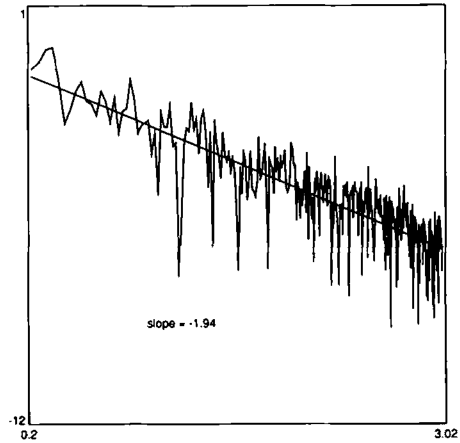

Fig. 7. Numerical estimate of the spectral density for the process $S_{[nt]}(x)/(2Dn)^{1/2}$, in the case of the Markov noise $2\beta \bmod [0, 1[$. The parameters are the same as in Fig. 2.

on ν of the spectral density turns out to be, with probability one, of the form $\mathscr{P}(\nu) \sim 1/\nu^2$. This "$1/\nu^2$ law" is usually considered as a typical feature of Brownian motion. In Fig. 7 we show the result of the above computation applied to arbitrary realizations of the Markov process, case (a), but similar results are obtained for the other stochastic processes described in the previous sections for which a central limit theorem seems to hold; different initializations $x \in \Omega$ have been taken. The bilogarithmic diagrams emphasize a power dependence of the spectral density $\mathscr{P}(\nu)$ on the frequency ν of type $1/\nu^2$, in accordance with the $1/\nu^2$ law characteristic of the Wiener process. This result constitutes a good indication of the validity of an IP for all the models considered here whenever the diffusion coefficient D is strictly positive.

5. CONCLUSIONS

In this paper we checked numerically the statistical properties of a model of modulated diffusion introduced by us in refs. 1–3; in particular, we investigated the central limit theorem and the invariance principle. As pointed out in ref. 3, our model describes the evolution of a Hamiltonian isochronous map near an invariant curve of given frequency whenever the variation of the frequency with the action is sufficiently small. The coupling of the action with the angle variable is replaced, in our model, by a deterministic or random perturbation of the frequency [see Eq. (2.1)]: what we get is therefore a sort of random standard map. In this light it becomes interesting to analyze the system in the whole range of the values of the perturbation parameter ε. We did that analytically for the diffusion coefficient and we effectively observed all the typical behaviors of the usual standard maps: quasilinear approximation for large perturbation parameters; existence of ballistic motion; superlinear regime at intermediate ε. The positivity of the diffusion coefficient is usually assumed as the principal indicator for the transport in Hamiltonian maps and the other statistical properties are not generally investigated, notably the invariance principle, which permits one to verify the existence of a full set of trajectories close to the realizations of Brownian motion. The work presented here goes in this direction and gives two main indications for future developments:

1. The numerical computations are very reliable and accurate, even with relatively poor statistics, and this encourages us to analyze more complicated models where the strong dependence of the random variables suggests asymptotic distributions other than the Gaussian one.

2. When the diffusion coefficient D is positive, the CLT and the IP hold no matter what the perturbation is. As explained in the Introduction, we expect this is related to the character of the noises considered: nevertheless, a rigorous proof seems very difficult to give. At the moment

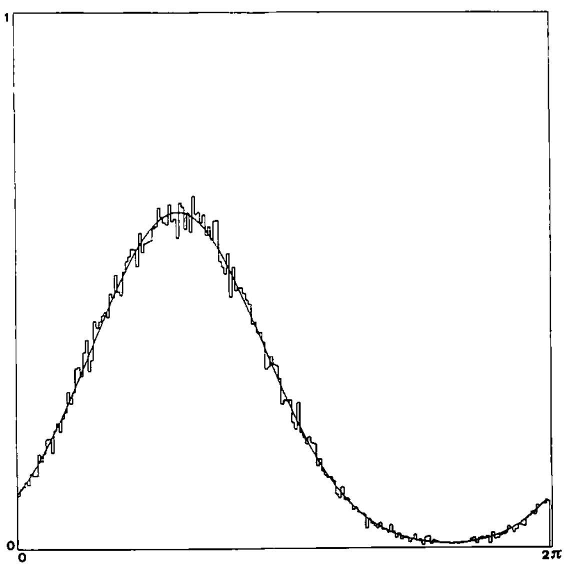

Fig. 8. Probability distribution after $n = 100$ iterations of the angle θ by assuming the initial condition $\theta_0 = \pi$ and a stochastic modulation of normal random variables, with $\omega/2\pi = (\sqrt{5} - 1)/2$ and $\varepsilon = 0.1$. The histogram is computed by 50,000 realizations of the perturbing process and compared with the solution $\rho(\theta, t)$ on the torus $[0, 2\pi[$ of the differential equation $\partial\rho/\partial t + \omega\, \partial\rho/\partial\theta = D\, \partial^2\rho/\partial\theta^2$, where $D = \mathbf{E}(f^2)/2$ and $\rho(\theta, 0) = \delta(\theta - \theta_0)$.

the only central limit property we are able to prove, on cases (a), (b), and (d), concerns the much simpler dynamics on the angular variable θ, as follows from

$$\frac{1}{\sqrt{n}\,K}\sum_{j=0}^{n-1} f(\alpha_j) \xrightarrow[n \to +\infty]{\text{in distribution}} \xi \tag{5.1}$$

with ξ a normal random variable and $K = [\mathbf{E}(f^2)]^{1/2}$. Of course, the convergence can be trivially deduced from standard CLT for i.i.d. stochastic processes (b) and (d), whereas it is less obvious for the Markov case (a)—see Fig. 8. Before concluding about the generality of our numerical results, one should look at a nontrivial example of a weakly random standard map with positive D and non-Gaussian distribution.

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